AD-F300218

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AD A126651

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TECHNICAL REPORT ARBRL-TR-02481

MAXIMUM LIKELIHOOD PROGRAM FOR SEQUENTIAL TESTING DOCUMENTATION

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March 1983

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TECHNICAL REPORT ARBRL-TR-02481	2. GOVT ACCESSION NO. AD-A126 651	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) MAXIMUM LIKELIHOOD PROGRAM FOR SEQUENTIAL TESTING DOCUMENTATION		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ann E. McKaig Jerry Thomas		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Ballistic Research Laboratory ATTN: DRDAR-BLB Aberdeen Proving Ground, MD 21005		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RDT&E 1L162618AH80
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Research & Development Command US Army Ballistic Research Laboratory (DRDAR-BL) Aberdeen Proving Ground, MD 21005		12. REPORT DATE March 1983
		13. NUMBER OF PAGES 64
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Maximum Likelihood Estimates
Newton-Raphson Procedure
Quantile Response
Sensitivity Testing

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(eal)
Quantal response problems encountered by the Ballistic Research Laboratory, such as armor plate penetration and primer testing evaluations, have been a subject of interest for some time. A computer program developed by A. R. DiDonato and M. P. Jarnagin, Jr., has been modified and installed on the Ballistic Research Laboratory's CDC computer. The program provides maximum likelihood estimates of the mean and standard deviation of the response distribution and the standard deviations of these estimates. The response function is assumed to have a cumulative (Cont'd)

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Item 20. (Cont'd)

normal distribution. A nonparametric algorithm is incorporated for the cases in which maximum likelihood estimates do not exist.

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I. INTRODUCTION

The Army has used sensitivity testing for many years, especially in the areas of primer testing and armor penetrating projectiles. In sensitivity testing, only one observation per sample is obtained. This observation is characterized by a quantal response, i.e., go or no-go, explosion or non-explosion, penetration or nonpenetration. The purpose of most sensitivity experiments is to determine the amount of stimuli needed in order to achieve a 50% probability of success.

For armor penetrating projectiles, the stimulus is velocity. So in sensitivity testing, the aim is to determine the V_{50}, i.e., the velocity needed to have a 50% probability of penetrating the armor.

There are many methods or schemes for collecting data in sensitivity tests. Some of the more popular ones are the Bruceton up-and-down method, the Langlie design, the Robbins-Monro design and the Alexander design. For a discussion of these designs see Reference 1.

As mentioned earlier, we are interested in estimating V_{50} and getting an estimate of its variability. The most commonly used technique of obtaining parameter estimates is to assume that the critical stimuli levels are normally distributed and to use maximum likelihood estimation (MLE) techniques to estimate μ (V_{50}) and σ. By critical stimuli levels, i.e., critical penetration velocities, it is meant those levels (velocities) at or above which a penetration would always take place. These estimates are not dependent upon the sensitivity design used to collect the data.

There are many computer programs that compute MLE's. However, most of these programs, at one time or another, have a convergence problem. That is, initial values of the estimates cannot be found that result in the program converging on estimates of the mean critical stimulus level, $\hat{\mu}$, and the standard deviation of the critical stimuli levels, $\hat{\sigma}$.

DiDonato and Jarnagin[2] used a Newton-Raphson procedure on transformed parameters and claim that it converges globally to the "best" estimates (when estimates exist), regardless of what initial values were used. Unique MLE's will exist as long as the data meet the following two restrictions: 1) minimum level, a_i, at which a success is observed < maximum level, b_j, at which a failure is observed, and 2) average a_i's (penetrations) > average b_j's (nonpenetrations). They documented a program written for an IBM computer using their procedure. DiDonato and Jarnagin's program was modified and installed on the Ballistic Research Laboratory's CDC computer.

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[1] D. Rothman, M.J. Alexander and J.M. Zimmerman, "The Design and Analysis of Sensitivity Experiments," NASA CR-62026-Vol. I, May 1965.

[2] A.R. DiDonato and M.P. Jarnagin, Jr., "Use of the Maximum Likelihood Method Under Quantal Responses for Estimating the Parameters of a Normal Distribution and its Application to an Armor Penetration Problem," Naval Weapons Laboratory, November 1972, NWL Technical Report TR-2846.

This report discusses the statistical theory needed to estimate the mean (V_{50}) and the standard deviation of the response distribution and to obtain the standard deviations of these estimates. A discussion of the DiDonato and Jarnagin computing procedure used to obtain these is provided. A nonparametric algorithm has been incorporated with the DiDonato and Jarnagin program to provide an estimate of the mean (V_{50}) and standard deviation of the response distribution when the data do not meet the requirements for the DiDonato and Jarnagin procedure. Examples are provided for each of these algorithms. A computer listing, the appropriate inputs and corresponding output of the program are included in the appendices. The reader who is more interested in learning to run the program rather than in the theory is referred directly to Appendix C.

II. THEORY

The principle of maximum likelihood was first introduced and extensively developed by R. A. Fisher. To aid in deriving the likelihood function, consider the results of an experiment (such as an armor plate penetration problem) in which the levels of the stimulus are not completely under control. There is generally only one observation per level, and the response for each level is either a "success" or a "failure." Call the probability of a successful response p_i, i=1, ..., n and the probability of a failure q_j, j=1, ..., m. Then it follows from elementary statistical concepts that the probability of the observed set of responses may be given by the product

$$L_* = p_1 p_2 \cdots p_n q_1 q_2 \cdots q_m .$$

Generally, if a total of n+m cases are tested and there are n successes recorded at stimulus levels $a_1, a_2, \ldots, a_n$ and m failures recorded at levels $b_1, b_2, \ldots, b_m$, and assuming the critical levels of stimulus are normally distributed, then the probability of occurrence of this set of events may be written in the more usual notation,

$$L_* = \prod_{i=1}^{n} p_i \prod_{j=1}^{m} q_j \tag{1}$$

where $$p_i \equiv p[(a_i-\mu)/\sigma] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(a_i-\mu)/\sigma} \exp(-t^2/2)dt, \tag{2}$$

and

$$q_j \equiv q[(b_j-\mu)/\sigma] = \frac{1}{\sqrt{2\pi}} \int_{(b_j-\mu)/\sigma}^{\infty} \exp(-t^2/2)dt. \tag{3}$$

The method of maximum likelihood now consists of choosing, as estimates of the unknown population values of μ and σ, those particular values that render L_* as large as possible. Since likelihood functions are products, and sums are usually more convenient to work with than products, the usual practice is to maximize the natural logarithm of the likelihood function rather than the likelihood function itself, i.e.,

$$L(\mu,\sigma) = \sum_{i=1}^{n} \ell n\ p_i + \sum_{j=1}^{m} \ell n\ q_j. \tag{4}$$

The logarithm of the likelihood function has its maximum at the same point as does the likelihood function.

The maximum-likelihood estimators of μ and σ are obtained by setting the partial derivatives of the likelihood function equal to zero, i.e.,

$$\frac{\partial L}{\partial \mu} = 0, \qquad \frac{\partial L}{\partial \sigma} = 0 \tag{5}$$

The likelihood equations can then be solved iteratively using some technique such as Newton-Raphson, to obtain estimates, $\hat{\mu}$ and $\hat{\sigma}$.

Previously it was assumed that there was a given set of observed responses, n successes at levels $a_1, a_2, \ldots, a_n$ and m failures at levels $b_1, b_2, \ldots, b_m$. Suppose that for a set of distinct levels of stimuli, ℓ_k, there is a statement associated with each level that it was either a "success" or a "failure." Consider the discrete random variable δ_k (k = 1,2,...,N), where N equals the number of levels observed and each δ_k may take on one of two possible values, i.e.,

$$\delta_k = \begin{cases} 0, \text{ if a failure occurs at level } \ell_k \\ 1, \text{ if a success occurs at level } \ell_k. \end{cases}$$

If the probabilities of success at the resulting levels are p_k and the probabilities of failure are $q_k = 1-p_k$, then the likelihood of observing some set of responses δ_k (k = 1,2,...,N) at levels ℓ_k (k = 1,2,...,N) is

$$L_{o*} = \prod_{k=1}^{N} p_k^{\ \delta_k} q_k^{(1-\delta_k)}, \tag{6}$$

and the natural logarithm of the likelihood function becomes,

$$L_o = \prod_{k=1}^{N} [\delta_k \ \ell n \ p_k + (1-\delta_k) \ \ell n \ q_k]. \tag{7}$$

It is this form of the likelihood function that will be used to construct the elements of the covariance matrix utilizing the theory of large sample distributions.[3] The variances of $\hat{\mu}$ and $\hat{\sigma}$ may be obtained using the relationships below between the expected values of the first and second-order partial derivatives of L_o (Appendix A),

$$-E\left(\frac{\partial^2 L_o}{\partial \mu^2}\right) = E\left(\frac{\partial L_o}{\partial \mu}\right)^2 = \frac{1}{\sigma^2} \sum_{k=1}^{N} \frac{z_k^2}{p_k q_k} \tag{8}$$

$$-E\left(\frac{\partial^2 L_o}{\partial \sigma^2}\right) = E\left(\frac{\partial L_o}{\partial \mu}\right)^2 = \frac{1}{\sigma^2} \sum_{k=1}^{N} \frac{v_k^2 z_k^2}{p_k q_k} \tag{9}$$

and

$$-E\left(\frac{\partial^2 L_o}{\partial \mu \partial \sigma}\right) = E\left(\frac{\partial L_o}{\partial \mu}\right)\left(\frac{\partial L_o}{\partial \sigma}\right) = \frac{1}{\sigma^2} \sum_{k=1}^{N} \frac{v_k z_k^2}{p_k q_k} . \tag{10}$$

Additionally,

$$-E\left(\frac{\partial^2 L_o}{\partial \mu^2}\right) = E\left(\frac{\partial L_o}{\partial \mu}\right)^2 = A_{\mu\mu} \tag{11}$$

$$-E\left(\frac{\partial^2 L_o}{\partial \mu \partial \sigma}\right) = E\left(\frac{\partial L_o}{\partial \mu}\right)\left(\frac{\partial L_o}{\partial \sigma}\right) = A_{\mu\sigma} \tag{12}$$

and

$$-E\left(\frac{\partial^2 L_o}{\partial \sigma^2}\right) = E\left(\frac{\partial L_o}{\partial \sigma}\right)^2 = A_{\sigma\sigma} \tag{13}$$

The covariance matrix for the distribution of the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ may be written as

$$A = \begin{pmatrix} A_{\mu\mu} & A_{\mu\sigma} \\ A_{\mu\sigma} & A_{\sigma\sigma} \end{pmatrix}^{-1} = \begin{pmatrix} A^{\mu\mu} & A^{\mu\sigma} \\ A^{\mu\sigma} & A^{\sigma\sigma} \end{pmatrix} \tag{14}$$

[3] A. Mood and F. Graybill, Introduction to the Theory of Statistics, McGraw-Hill Book Company, Inc., 1963, pp. 235-239.

where $A^{\mu\mu}$ and $A^{\sigma\sigma}$ are the asymptotic variances of $\hat{\mu}$ and $\hat{\sigma}$, respectively, and $A^{\mu\sigma}$ is the asymptotic covariance of $\hat{\mu}$ and $\hat{\sigma}$. It should be noted that $A^{\mu\mu}$ is the asymptotic variance of the estimate of V_{50} while $A^{\sigma\sigma}$ is the asymptotic variance of the standard deviation, $\hat{\sigma}$, of the response distribution.

For convenience of notation, equations (8) - (13) may be rewritten. Recall that $k = 1,2,\ldots, N$, and divide N into n successes (penetrations) and m failures (nonpenetrations), so

$$N = n + m.$$

The n successes recorded at stimulus levels $a_1, a_2, \ldots, a_n$ and m failures recorded at levels $b_1, b_2, \ldots, b_m$ may be rewritten as

$$s_i = (a_i - \mu)/\sigma, \text{ where } i = 1,2,\ldots, n \tag{15}$$

and

$$t_j = (b_j - \mu)/\sigma, \text{ where } j = 1,2,\ldots, m, \tag{16}$$

which are their standardized normal equivalents. The s_i are associated with successes and the t_j are associated with failures.[1]

Also,

$$x_i \equiv z(s_i) = (1/\sqrt{\;} \text{[illegible]} (-s_i^2/2), \text{ when a success is observed} \tag{17}$$

and

$$y_j \equiv z(t_j) = (1/\sqrt{2\pi}) \exp \text{[illegible]}_j /2), \text{ when a failure is observed.} \tag{18}$$

Equations (8) - (10), which rep[illegible]t the expected values of the second-order partial derivatives taken ove[illegible] entire set of observations, can each be partitioned into the sum ove[illegible] set of successes plus the sum over the set of failures.

The coefficients of $\hat{\sigma}^2$ [illegible]$_{\mu\mu}$, $A_{\mu\sigma}$, $A_{\sigma\sigma}$] may be expressed as

$$\hat{\sigma}^2 A_{\mu\mu} \text{[illegible]}_1 \frac{x_i^2}{p_i q_i} + \sum_{j=1}^{m} \frac{y_j^2}{p_j q_j} \tag{19}$$

$$\hat{\sigma}^2 A_{\mu\sigma} = \sum_{i=1}^{n} \frac{s_i x_i^2}{p_i q_i} + \sum_{j=1}^{m} \frac{t_j y_j^2}{p_j q_j} \ . \tag{20}$$

and

$$\hat{\sigma}^2 A_{\sigma\sigma} = \sum_{i=1}^{n} \frac{s_i^2 x_i^2}{p_i q_i} + \sum_{j=1}^{m} \frac{t_j^2 y_j^2}{p_j q_j} \quad . \tag{21}$$

III. PROGRAM DESCRIPTION

This section first presents an overview of the DiDonato-Jarnagin procedure, including: i) the reparameterization of the likelihood function, ii) the actual procedure used to obtain $\hat{\mu}$ and $\hat{\sigma}$, and iii) a description of the output sheet. Should a more detailed mathematical description of the analyses be required, the reader is referred to the DiDonato and Jarnagin study.[2]

A nonparametric algorithm which has been incorporated into the original program and its associated output sheet are also summarized. This procedure closely resembles a procedure presently utilized by the Materiel Testing Directorate, Aberdeen Proving Ground, Maryland.

The complete program is available for use on the BRL CDC computer system. Both the DiDonato-Jarnagin procedure and the nonparametric algorithm are written in FORTRAN IV.

A. DiDonato-Jarnagin Procedure

1. Likelihood Function. Recalling the likelihood function for an observed set of n successes at stimulus levels $a_1, a_2, \ldots, a_n$ and m failures at stimulus levels $b_1, b_2, \ldots, b_m$, we have

$$L(\mu,\sigma) = \sum_{i=1}^{n} \ell n\, p_i + \sum_{j=1}^{m} \ell n\, q_j . \tag{22}$$

DiDonato and Jarnagin then apply the following transformations to replace the parameters μ and σ with new parameters α and β.

$$\alpha = \mu/\sigma \qquad [\mu = \alpha/\beta] \tag{23}$$

$$\beta = 1/\sigma > 0 \quad [\sigma = 1/\beta]. \tag{24}$$

In addition, the original stimulus levels are reexpressed as,

$$s_i = a_i\beta - \alpha \tag{25}$$

$$t_j = b_j\beta - \alpha . \tag{26}$$

so that p_i and q_j are transformed in terms of the new parameters to

$$p_i = p(s_i) = \int_{-\infty}^{s_i} z(t)dt \tag{27}$$

$$q_j = q(t_j) = \int_{t_j}^{\infty} z(t)dt \tag{28}$$

where $$z(t) = (1/\sqrt{2\pi}) \exp(-t^2/2) \tag{29}$$

For ease in computing the partial derivatives of the logarithm of the likelihood function, the computations of the needed partial derivatives of p_i and q_j are included as Appendix B. The partial derivatives of the logarithm of the likelihood function, L, used in the Newton-Raphson procedure are

$$L\alpha = \frac{\partial L}{\partial \alpha} = \sum_{j=1}^{m} (y_j/q_j) - \sum_{i=1}^{n} (x_i/p_i) \tag{30}$$

$$L\beta = \frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} a_i(x_i/p_i) - \sum_{j=1}^{m} b_j(y_j/q_j) \tag{31}$$

$$L\alpha\alpha = \frac{\partial^2 L}{\partial \alpha^2} = - \sum_{j=1}^{m} (y_j/q_j)[(y_j/q_j) - t_j] - \sum_{i=1}^{n} (x_i/p_i)[(x_i/p_i) + s_i] \tag{32}$$

$$L\alpha\beta = \frac{\partial^2 L}{\partial \alpha \partial \beta} = \sum_{j=1}^{m} b_j(y_j/q_j)[(y_j/q_j) - t_j] + \sum_{i=1}^{n} a_i(x_i/p_i)[(x_i/p_i) + s_i] \tag{33}$$

$$L\beta\beta = \frac{\partial^2 L}{\partial \beta^2} = - \sum_{j=1}^{m} b_j^2(y_j/q_j)[(y_j/q_j) - t_j] - \sum_{i=1}^{n} a_i^2(x_i/p_i)[(x_i/p_i) + s_i] \tag{34}$$

where $$x_i \equiv z(s_i) = (1/\sqrt{2\pi}) \exp(-s_i^2/2) \tag{35}$$

$$y_j \equiv z(t_j) = (1/\sqrt{2\pi}) \exp(-t_j^2/2). \tag{36}$$

The $\{a_i$ = stimulus level at which there is a response, penetration$\}$ and $\{b_j$ = stimulus level at which there is a nonresponse, nonpenetration$\}$, where $i=1, \ldots, n$, $j=1, \ldots, m$, represent input to the program. The quantities a_i and b_j may take on any arbitrary real values although they are almost always positive. The standard deviation, σ, and its reciprocal, β, are by nature always positive.

In their study, DiDonato and Jarnagin point out that to insure the existence of a unique point at which L attains a maximum, the a_i and b_j must satisfy certain necessary and sufficient conditions, i.e.,

$$a_{min} < b_{max} \qquad \text{[zone of mixed results]} \tag{37}$$

where $a_{min} = \min(a_i)$ and $b_{max} = \max(b_j)$,

and

$$\frac{1}{m} \sum b_j < \frac{1}{n} \sum a_i . \tag{38}$$

2. Computing Procedure. The DiDonato-Jarnagin procedure first insures that the necessary and sufficient conditions indicated in (37) and (38) are satisfied by the input. If either condition is not satisfied, then no maximum likelihood estimates exist for μ and σ and the program is routed to the nonparametric algorithm. If both conditions are satisfied, an initial point (α_o,β_o) is obtained (unless the user supplies initial estimates, Appendix C) using the following relationships,

$$\alpha_o = \mu_o/\sigma_o = \frac{1}{2} \left(\frac{1}{n} \sum_{i=1}^{n} a_i + \frac{1}{m} \sum_{j=1}^{m} b_j\right) / \left[\frac{1}{n+m} \left(\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{m} b_j^2\right) - v^2\right]^{1/2} \tag{39}$$

$$\beta_o = 1/\sigma_o = \left[\frac{1}{n+m} \left(\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{m} b_j^2\right) - v^2\right]^{1/2} \tag{40}$$

where

$$v = \frac{1}{n+m} \left(\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j\right). \tag{41}$$

After choosing an initial point, the ordinary Newton-Raphson (N-R) procedure is applied in order to reduce $L\alpha$ and $L\beta$ to zero simultaneously and produce the N-R increments $\Delta\alpha$ and $\Delta\beta$ and a new point (α_1,β_1). The two iterative equations utilized by the estimation scheme are

$$\Delta\alpha L\alpha\alpha + \Delta\beta L\alpha\beta = -L\alpha \tag{42}$$

$$\Delta\alpha L\alpha\beta + \Delta\beta L\beta\beta = -L\beta. \tag{43}$$

Solving this linear system of equations yields the increments $\Delta\alpha$ and $\Delta\beta$ as follows:

$$\Delta\alpha = (L\beta \; L\alpha\beta - L\alpha \; L\beta\beta)/\Delta \quad (44)$$

$$\Delta\beta = (L\alpha \; L\alpha\beta - L\beta \; L\alpha\alpha)/\Delta \quad (45)$$

where $$\Delta \equiv L\alpha\alpha \; L\beta\beta - L^2\alpha\beta > 0. \quad (46)$$

The increments $\Delta\alpha$ and $\Delta\beta$ are the changes in the old values of α and β calculated at each iteration of the N-R procedure.

If a sequence, (α_k,β_k), is determined by iteration and the α_k and β_k denote the k^{th} approximations to $\hat{\alpha}$ and $\hat{\beta}$, then the iterations are terminated when

$$|\Delta\alpha_k| < \varepsilon_1\alpha_k \quad \text{and} \quad |\Delta\beta_k| < \varepsilon_2\beta_2. \quad (47)$$

The parameters ε_1 and ε_2 are defined as follows:

$$\varepsilon_1 = 2.5 \times 10^{-4} = 1/2 \; \varepsilon_2. \quad (48)$$

At this point, additional notation is introduced in order to rewrite equations (19) - (21) and (30) - (34) as actually used in the DiDonato-Jarnagin procedure.

Referring to the parameter input description in Appendix C, it can be seen that only those a_i which are different are listed along with an integer $n(i)$ which denotes the number of times a_i appears as input. The integer $m(j)$ denotes the number of times each different b_j appears as input. Noting that the expressions for $L\alpha$, $L\beta$, $L\alpha\alpha$, $L\alpha\beta$, $L\beta\beta$ are linear sums, quantities such as (x_i/p_i) and (y_j/q_j) need only be computed for the different a_i or b_j, then multiplied by $n(i)$ or $m(j)$, respectively.

If the k^{th} different a_i is denoted by $a(k)$, the r^{th} different b_j is denoted by $b(r)$, and K and R denote the total number of different a_i and b_j, respectively, then

$$n = \sum_{k=1}^{K} n(k), \qquad m = \sum_{r=1}^{R} m(r) \quad (49)$$

and n and m have their usual meaning. The equations referenced above can now be written as:

$$L\alpha = \sum_{r=1}^{R} m(r)(y_r/q_r) - \sum_{k=1}^{K} n(k)(x_k/p_k) \tag{50}$$

$$L\beta = \sum_{k=1}^{K} n(k)a(k)(x_k/p_k) - \sum_{r=1}^{R} m(r)b(r)(y_r/q_r) \tag{51}$$

$$L\alpha\alpha = - \sum_{r=1}^{R} m(r)(y_r/q_r)(y_r/q_r - t_r)$$

$$- \sum_{k=1}^{K} n(k)(x_k/p_k)(x_k/p_k + s_k) \tag{52}$$

$$L\alpha\beta = \sum_{k=1}^{K} n(k)a(k)(x_k/p_k)(x_k/p_k + s_k)$$

$$+ \sum_{r=1}^{R} m(r)b(r)(y_r/q_r)(y_r/q_r - t_r) \tag{53}$$

$$L\beta\beta = - \sum_{r=1}^{R} m(r)b^2(r)(y_r/q_r)(y_r/q_r - t_r)$$

$$- \sum_{k=1}^{K} n(k)a^2(k)(x_k/p_k)(x_k/p_k + s_k) \tag{54}$$

$$\hat{\sigma}^2 A\mu\mu = \sum_{k=1}^{K} n(k)(x_k/p_k)(x_k/q_k)$$

$$+ \sum_{r=1}^{R} m(r)(y_r/p_r)(y_r/q_r) \tag{55}$$

$$\hat{\sigma}^2 A\mu\sigma = \sum_{k=1}^{K} n(k)s_k(x_k/p_k)(x_k/q_k)$$

$$+ \sum_{r=1}^{R} m(r)t_r(y_r/p_r)(y_r/q_r) \tag{56}$$

$$\hat{\sigma}^2 A\sigma\sigma = \sum_{k=1}^{K} n(k) s_k^{\,2} (x_k/p_k)(x_k/q_k)$$

$$+ \sum_{r=1}^{R} m(r) t_r^{\,2} (y_r/p_r)(y_r/q_r), \qquad (57)$$

where $$s_k = a(k)\ \beta\text{-}\alpha \qquad (58)$$

$$t_r = b(r)\ \beta\text{-}\alpha\ . \qquad (59)$$

3. Output Description. The output for examples 1 and 3 (Section 4) having MLE's generated under this procedure is shown in Appendix D Beginning in the upper left-hand corner, each output sheet lists a sequence of alphanumeric characters which may indicate a title, identification number, etc. This is followed by a listing of the different a_i and b_j and the number of times each value occurs. The maximum likelihood estimates, $\hat{\mu}$ and $\hat{\sigma}$, identified as 'MU' and 'SIGMA,' respectively, are listed toward the top center. 'MU,' or $\hat{\mu}$, is the estimate of V_{50} while 'SIGMA,' or $\hat{\sigma}$, is the estimate of the standard deviation of the response distribution. Directly below the estimates, the elements of the covariance matrix are recorded, followed by $\hat{\sigma}_\mu$ and $\hat{\sigma}_\sigma$ identified as the 'STD DEV MU' and 'STD DEV SIGMA,' respectively. These quantities are followed by the initial approximations of α_o ('ALPHØ'), β_o ('BETAØ'), μ_o ('MUØ'), and σ_o ('SIGMAØ'). The Newton-Raphson increments at each iteration, $\Delta\alpha$, $\Delta\beta$, written as 'DELTA ALPHA' and 'DELTA BETA,' as well as the associated value of L_* are listed next. The last line lists the final estimates of the likelihood function L_*, the determinant Δ, α, and β and are identified as 'MAXIMUM,' 'DELTA,' 'ALPHA,' and 'BETA,' respectively.

B. Nonparametric Algorithm

1. Computing Algorithm. As previously mentioned, this algorithm is initiated should either of the necessary and sufficient conditions imposed upon the input data not be satisfied. This routine will compute a maximum of three (3) estimates for the mean and standard deviation using a method of averaging particular levels of response and nonresponse. To facilitate execution of the algorithm, the original input arrays have been expanded to include all a_i's and b_j's, including replicated values. Each array is rank ordered from the smallest to the largest level.

The first estimate of the mean is obtained by averaging the lowest value of a response and the highest value of a nonresponse. Then the two lowest responses and the two largest nonresponses are averaged to obtain a second

estimate for the mean. This iterative procedure may be continued until all even number combinations of the a_i's and b_j's are exhausted. However, it has been elected to allow the algorithm to generate only a possible maximum of three (3) estimates (requiring a total of six (6) input levels). At each iteration an estimate of the standard deviation of the distribution is calculated as follows:

$$1/2 \times [\text{range between levels}] \tag{60}$$

where 'range between levels' is defined as the difference between

$$\max [B\emptyset(MTOTB),\ A\emptyset(I)] - \min [B\emptyset(J),\ A\emptyset(1)] \tag{61}$$

and $A\emptyset(1)$ = smallest response value of input array (62)

$A\emptyset(I)$ = response value at k^{th} iteration, k=1, ...,3 (63)

$B\emptyset(J)$ = nonresponse value at k^{th} iteration, k=1, ...,3 (64)

$B\emptyset(MTOTB)$ = largest nonresponse value of input array. (65)

2. Output Description. The output generated for example 2 (Section 4) is shown in Appendix D. At the top of the output sheet, the program prints out a description of the first necessary and sufficient condition that is not satisfied. Directly under this is an ordered listing of the input arrays of responses and nonresponses, including replications. Finally, for each iteration, the estimate number and the number of rounds required for each estimate of μ and σ followed by the values for $\hat{\mu}$ and $\hat{\sigma}$, written as 'MU' and 'SIGMA,' respectively, are shown.

IV. APPLICATIONS (EXAMPLES)

The three cases used for illustration are taken from armor plate penetration studies. The actual output of the computer program for each of the examples is included as Appendix D. The data set for the first example satisfied the necessary and sufficient conditions outlined in Section III; therefore, the output will be generated from the DiDonato-Jarnagin procedure. Example 2 is a 'no zone of mixed results' case which utilized the nonparametric algorithm. The third example satisfied both conditions required for MLE's to exist for $\hat{\mu}$ and $\hat{\sigma}$, yet diverged using computer generated starting values in a different estimation program. This case is shown for two different sets of starting values for which the results show convergence.

For each example the initial estimates were generated by the program; in addition, the third example was run a second time with user supplied starting values for α_o and β_o.

EXAMPLE 1

In firing 10 rounds at a given armor plate the following observations were recorded:

VELOCITY (m/s)	RESPONSE
1008	Penetration
976	Nonpenetration
989	Penetration
973	Penetration
947	Penetration
943	Penetration
924	Nonpenetration
931	Nonpenetration
960	Penetration
942	Nonpenetration

Initial estimates computed were α_o = 37.567 m/s (μ_o = 956.625 m/s), β_o = .039 m/s (σ_o = 25.464 m/s) The N-R iteration procedure was then applied 4 times to generate final estimates of $\hat{\alpha}$ = 32.123 m/s ($\hat{\mu}$ 948.823 m/s), $\hat{\beta}$ = .034 m/s ($\hat{\sigma}$ = 29.537 m/s). Employing the solutions obtained, the approximate asymptotic variances of $\hat{\mu}$ and $\hat{\sigma}$ were obtained. The covariance matrix is given by

$$\begin{matrix} 183.506 & -50.653 \\ -50.653 & 358.946 \end{matrix} \; .$$

Thus, $\sigma^2_{\hat{\mu}}$ = 183.506 m/s ($\sigma_{\hat{\mu}}$ = 13.546 m/s) and $\sigma^2_{\hat{\sigma}}$ = 358.946 m/s ($\sigma_{\hat{\sigma}}$ = 18.946 m/s).

EXAMPLE 2

Seven rounds of a given projectile were fired at a given armor plate. The following observations were recorded:

VELOCITY (m/s)	RESPONSE
944	Nonpenetration
973	Penetration
961	Nonpenetration
982	Penetration
966	Penetration
949	Nonpenetration
970	Penetration

Once the data were sorted, it was clearly seen that there was 'no zone of mixed results.' The program then routed to the nonparametric algorithm. To illustrate the method, let's calculate the second estimate of MU ($\hat{\mu}$) and SIGMA ($\hat{\sigma}$). The second estimate required a total of 4 levels or rounds (2 lowest responses, 2 highest nonresponses). From the sorted input lists, the two lowest responses were 966 m/s and 970 m/s; the two highest nonresponses were 949 m/s and 961 m/s. From the routine,

MUØ = MUØ + (A(2) + B(2))/2.0

Estimate 1 for MU = 963.5, so

MUØ = 963.5 + (970 + 949)/2.0

MUØ = 1923

W = 2.0 (W = KOUNT; KOUNT = iteration no.)

MU = MUØ/W = 1923/2.0 = 961.5 m/s

SIGMA = (RRANGE - SRANGE)/2.0

Where RRANGE = MAX(BØ(MTOTL), AØ(I))

= MAX(961, 970)

= 970

SRANGE = MIN(BØ(J), AØ(1))

= MIN(949, 966)

= 949

Thus, SIGMA = (970 - 949)/2.0

= 10.5 m/s

EXAMPLE 3

Again, 10 rounds were fired at a given armor plate and yielded the following results:

VELOCITY (m/s)	RESPONSE
1295	Nonpenetration
1296	Penetration
1298	Nonpenetration
1301	Nonpenetration
1303	Nonpenetration
1304	Nonpenetration
1307	Nonpenetration
1310	Penetration
1311	Penetration
1314	Nonpenetration

a) Initial estimates generated by the program were α_o = 210.224 m/s (μ_o = 1304.405 m/s), β_o = .161 m/s (σ_o = 6.205 m/s). Applying the N-R scheme 3 times produced final estimates of $\hat{\alpha}$ = 50.657 m/s ($\hat{\mu}$ = 1317.893 m/s), $\hat{\beta}$ = .038 m/s ($\hat{\sigma}$ = 26.016 m/s). The following asymptotic variances were obtained:

$\sigma^2_{\hat{\mu}}$ = 690.476 m/s ($\sigma_{\hat{\mu}}$ = 26.277 m/s)

and

$\sigma^2_{\hat{\sigma}}$ = 2142.252 m/s ($\sigma_{\hat{\sigma}}$ = 46.283 m/s).

b) Initial approximations of α_o = 26.000 m/s, β_o = .006 m/s were input by the user. These values generated initial values for μ_o and σ_o of 4333.333 m/s and 166.667 m/s, respectively. The N-R procedure had to be applied 5 times. Final estimates for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mu}$, $\hat{\sigma}$ were 50.658 m/s, .038 m/s, 1317.892 m/s, and 26.016 m/s, respectively. Applying these solutions yielded the following asymptotic variances for $\hat{\mu}$ and $\hat{\sigma}$;

$\sigma^2_{\hat{\mu}}$ = 690.443 m/s ($\sigma_{\hat{\mu}}$ = 26.276 m/s)

and

$\sigma^2_{\hat{\sigma}}$ = 2142.057 m/s ($\sigma_{\hat{\sigma}}$ = 46.282 m/s).

The above results clearly show convergence.

V. SUMMARY

For problems of a quantal response nature, a discussion of the maximum likelihood estimation theory and large-sample distribution theory required to obtain the estimates of the mean (V_{50}) and standard deviation of a cumulative normal response function has been given.

The A. R. DiDonato and M. P. Jarnagin, Jr. maximum likelihood estimation program has been modified and installed on the Ballistic Research Laboratory's CDC computer. Instructions needed to properly execute the program have been provided. This program has successfully converged to provide maximum likelihood estimates for all data sets tried by the authors. A nonparametric algorithm is also available for those cases where maximum likelihood estimates do not exist.

VI. ACKNOWLEDGEMENTS

The authors wish to express their appreciation to the following individuals of the Experimental Design and Analysis Branch of the BRL: William E. Baker and Barry A. Bodt for their programming technical support and Jill H. Smith and Jock O. Grynovicki for reviewing the text and providing technical assistance.

APPENDIX A. EXPECTED VALUES NEEDED FOR ASYMPTOTIC VARIANCES OF μ AND σ

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The variances of $\hat{\mu}$ and $\hat{\sigma}$ may be obtained in accordance with maximum likelihood theory and large-sample theory from the expected values of the second-order partial derivatives of the function L_o,

$$L_o = \ell n\ L_{o*} = \sum_{k=1}^{N} [\delta_k\ \ell n\ p_k + (1 - \delta_k)\ \ell n\ q_k]. \tag{A1}$$

Use of the following general relationship will simplify the computation of the expected values,

$$E\left\{\left[\frac{\partial L_o}{\partial \theta}\right]^2\right\} = -E\left[\frac{\partial^2 L_o}{\partial \theta^2}\right], \tag{A2}$$

where the partial derivative is taken with respect to some parameter θ. Here, θ would be equivalent to μ and/or σ.

$$\text{Define } z_k = (1/\sqrt{2\pi})\ \exp^{-v_k^2/2}, \qquad \text{where } v_k = (\ell_k - \mu)/\sigma. \tag{A3}$$

and $$p_k = \int_{-\infty}^{v_k} (1/\sqrt{2\pi}) \exp^{-t^2/2}\ dt = 1 - q_k$$

Then,

$$\frac{\partial p_k}{\partial \mu} = \frac{-z_k}{\sigma}\ ; \quad \frac{\partial q_k}{\partial \mu} = \frac{z_k}{\sigma}\ ; \quad \frac{\partial p_k}{\partial \sigma} = \frac{-v_k z_k}{\sigma} \tag{A4}$$

and

$$\frac{\partial q_k}{\partial \sigma} = \frac{v_k z_k}{\sigma}\ .$$

The maximum likelihood equations with respect to μ and σ may be written as

$$\frac{\partial L_o}{\partial \mu} = \sum_{k=1}^{N} \left[\delta_k \frac{\partial}{\partial \mu} \ln p_k + (1-\delta_k) \frac{\partial}{\partial \mu} \ln q_k \right]$$

$$= \sum_{k=1}^{N} \left[\delta_k \cdot \frac{1}{p_k} \cdot \frac{-z_k}{\sigma} + (1-\delta_k) \cdot \frac{1}{q_k} \cdot \frac{z_k}{\sigma} \right]$$

$$\frac{\partial L_o}{\partial \mu} = \frac{1}{\sigma} \sum_{k=1}^{N} \left[(1-\delta_k) \frac{z_k}{q_k} - \delta_k \frac{z_k}{p_k} \right] \tag{A5}$$

$$\frac{\partial L_o}{\partial \sigma} = \sum_{k=1}^{N} \left[\delta_k \frac{\partial}{\partial \sigma} \ln p_k + (1-\delta_k) \frac{\partial}{\partial \sigma} \ln q_k \right]$$

$$= \sum_{k=1}^{N} \left[\delta_k \cdot \frac{1}{p_k} \cdot \frac{-v_k z_k}{\sigma} + (1-\delta_k) \cdot \frac{1}{q_k} \cdot \frac{v_k z_k}{\sigma} \right]$$

$$\frac{\partial L_o}{\partial \sigma} = \frac{1}{\sigma} \sum_{k=1}^{N} \left[(1-\delta_k) \frac{v_k z_k}{q_k} - \delta_k \frac{v_k z_k}{p_k} \right] \tag{A6}$$

Now, consider the fact that $E[g(\delta_k)] = \sum_{k=1}^{N} g(\delta_k) f(\delta_k)$, where $E(\delta_k) = f(1) = p_k$ and $E(1-\delta_k) = f(0) = q_k$; then the expected values of equations (A5) and (A6) become

$$E\left(\frac{\partial L_o}{\partial \mu} \right) = \frac{1}{\sigma} \sum_{k=1}^{N} \left[(1-0) \frac{z_k}{q_k} \cdot q_k - (1) \frac{z_k}{p_k} \cdot p_k \right]$$

$$= \frac{1}{\sigma} \sum_{k=1}^{N} [z_k - z_k]$$

$$E\left(\frac{\partial L_o}{\partial \mu} \right) = 0. \tag{A7}$$

$$E\left(\frac{\partial L_o}{\partial \sigma}\right) = \frac{1}{\sigma}\sum_{k=1}^{N}\left[(1-0)\frac{v_k z_k}{q_k}\cdot q_k - (1)\frac{v_k z_k}{p_k}\cdot p_k\right]$$

$$= \frac{1}{\sigma}\sum_{k=1}^{N}[v_k z_k - \text{[illegible]}]$$

$$E\left(\frac{\partial L_o}{\partial \sigma}\right) = 0. \tag{A8}$$

Using equations (A2) and (A5) - (A6) and the definitions developed in the previous paragraph, the expected values of the second-order partial derivatives of L_o may be easily computed as follows:

$$-E\left[\frac{\partial^2 L_o}{\partial \mu^2}\right] = E\left[\frac{\partial L_o}{\partial \mu}\right]^2$$

$$= E\left[\frac{1}{\sigma}\sum_{k=1}^{N}\left[(1-\delta_k)\frac{z_k}{q_k} - \delta_k\frac{z_k}{p_k}\right]\right]^2$$

$$= E\ \frac{1}{\sigma^2}\sum_{k=1}^{N}\left[(1-\delta_k)^2\frac{z_k^2}{q_k^2} + \delta_k^2\frac{z_k^2}{p_k^2}\right.$$

$$\left. - 2\delta_k(1-\delta_k)\frac{z_k^2}{p_k q_k}\right]$$

$$= \frac{1}{\sigma^2}\sum_{k=1}^{N}\left[(1-0)^2\frac{z_k^2}{q_k^2}\cdot q_k + (1)^2\frac{z_k^2}{p_k^2}\cdot p_k\right.$$

$$\left. - 0\right]$$

$$-E\left[\frac{\partial^2 L_o}{\partial \mu^2}\right] = \frac{1}{\sigma^2}\left[\sum_{k=1}^{N}\frac{z_k^2}{q_k} + \sum_{k=1}^{N}\frac{z_k^2}{p_k}\right] \tag{A9}$$

$$-E\left[\frac{\partial^2 L_o}{\partial\sigma^2}\right] = E\left\{\left[\frac{\partial L_o}{\partial\sigma}\right]^2\right\}$$

$$= E\left\{\left[\frac{1}{\sigma}\sum_{k=1}^{N}\left[(1-\delta_k)\frac{v_k z_k}{q_k} - \delta_k\frac{v_k z_k}{p_k}\right]\right]^2\right\}$$

$$= E\left\{\frac{1}{\sigma^2}\sum_{k=1}^{N}\left[(1-\delta_k)^2\frac{v_k^2 z_k^2}{q_k^2} + \delta_k^2\frac{v_k^2 z_k^2}{p_k^2}\right.\right.$$

$$\left.\left. - 2\delta_k(1-\delta_k)\frac{v_k^2 z_k^2}{p_k q_k}\right]\right\}$$

$$= \frac{1}{\sigma^2}\sum_{k=1}^{N}\left[(1-0)^2\frac{v_k^2 z_k^2}{q_k^2}\cdot q_k + (1)^2\frac{v_k^2 z_k^2}{p_k^2}\cdot p_k\right.$$

$$\left. - 0\right]$$

$$-E\left[\frac{\partial^2 L_o}{\partial\sigma^2}\right] = \frac{1}{\sigma^2}\left[\sum_{k=1}^{N}\frac{v_k^2 z_k^2}{q_k} + \sum_{k=1}^{N}\frac{v_k^2 z_k^2}{p_k}\right] \tag{A10}$$

$$-E\left[\frac{\partial^2 L_o}{\partial\mu\partial\sigma}\right] = E\left\{\left(\frac{\partial L_o}{\partial\mu}\right)\left(\frac{\partial L_o}{\partial\sigma}\right)\right\}$$

$$= E\left\{\frac{1}{\sigma}\sum_{k=1}^{N}\left[(1-\delta_k)\frac{z_k}{q_k} - \delta_k\frac{z_k}{p_k}\right]\right.$$

$$\left.\cdot\frac{1}{\sigma}\sum_{k=1}^{N}\left[(1-\delta_k)\frac{v_k z_k}{q_k} - \delta_k\frac{v_k z_k}{p_k}\right]\right\}$$

$$= E\left\{\frac{1}{\sigma^2}\sum_{k=1}^{N}\left[(1-\delta_k)^2\frac{v_k z_k^2}{q_k^2} + \delta_k^2\frac{v_k z_k^2}{p_k^2}\right.\right.$$

$$\left.\left. - 2\delta_k(1-\delta_k)\frac{v_k z_k^2}{p_k q_k}\right]\right\}$$

$$= \frac{1}{\sigma^2}\sum_{k=1}^{N}\left[(1-0)^2\frac{v_k z_k^2}{q_k^2}\cdot q_k + (1)^2\frac{v_k z_k^2}{p_k^2}\cdot p_k\right.$$

$$\left. - 0\right]$$

$$-E\left[\frac{\partial^2 L_o}{\partial\mu\partial\sigma}\right] = \frac{1}{\sigma^2}\left[\sum_{k=1}^{N}\frac{v_k z_k^2}{q_k} + \sum_{k=1}^{N}\frac{v_k z_k^2}{p_k}\right] \tag{A11}$$

Therefore, equations (A9) - (A11) may be rewritten as

$$-E\left[\frac{\partial^2 L_o}{\partial\mu^2}\right] = \frac{1}{\sigma^2}\sum_{k=1}^{N}\frac{z_k^2}{p_k q_k}\ , \tag{A12}$$

$$-E\left[\frac{\partial^2 L_o}{\partial\sigma^2}\right] = \frac{1}{\sigma^2}\sum_{k=1}^{N}\frac{v_k^2 z_k^2}{p_k q_k}\ , \tag{A13}$$

and

$$-E\left[\frac{\partial^2 L_o}{\partial\mu\partial\sigma}\right] = \frac{1}{\sigma^2}\sum_{k=1}^{N}\frac{v_k z_k^2}{p_k q_k}\ . \tag{A14}$$

APPENDIX B. PARTIAL DERIVATIVES NEEDED FOR NEWTON-RAPHSON PROCEDURE

APPENDIX B. PARTIAL DERIVATIVES NEEDED FOR NEWTON-RAPHSON PROCEDURE

We wish to compute the partial derivatives of the p_i and q_j as an aid in computing the partial derivatives of the function,

$$L = \ell n L_* = \sum_{i=1}^{n} \ell n\, p_i + \sum_{j=1}^{m} \ell n\, q_j . \tag{B1}$$

Let the p_i and q_j be expressed in terms of the new parameters as follows:

$$p_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a_i\beta-\alpha} \exp[-t^2/2]dt, \quad \text{recall } s_i = a_i\beta-\alpha \tag{B2}$$

and

$$q_j = \frac{1}{\sqrt{2\pi}} \int_{b_j\beta-\alpha}^{\infty} \exp[-t^2/2]dt, \quad \text{recall } t_j = b_j\beta-\alpha \tag{B3}$$

Also,

$$x_i \equiv (1/\sqrt{2\pi})\ \exp(-s_i^{\,2}/2) \tag{B4}$$

and

$$y_j \equiv (1/\sqrt{2\pi})\ \exp(-t_j^{\,2}/2) \tag{B5}$$

Let

$$\frac{\partial p_i}{\partial \alpha} = (-1/\sqrt{2\pi})\ \exp[-(a_i\beta-\alpha)^2/2] = -x_i \tag{B6}$$

then,

$$\frac{\partial^2 p_i}{\partial \alpha^2} = \frac{\partial(-x_i)}{\partial \alpha} = -\frac{(a_i\beta-\alpha)}{\sqrt{2\pi}}\ \exp[-(a_i\beta-\alpha)^2/2]$$

$$\frac{\partial^2 p_i}{\partial \alpha^2} = -s_i x_i \tag{B7}$$

$$\frac{\partial^2 p_i}{\partial \alpha \partial \beta} = \frac{\partial(-x_i)}{\partial \beta} = \frac{a_i(a_i\beta-\alpha)}{\sqrt{2\pi}}\ \exp[-(a_i\beta-\alpha)^2/2]$$

$$\frac{\partial^2 p_i}{\partial \alpha \partial \beta} = a_i s_i x_i , \tag{B8}$$

$$\frac{\partial p_i}{\partial \beta} = \frac{a_i}{\sqrt{2\pi}} \exp[-(a_i\beta-\alpha)^2/2] = a_i x_i, \tag{B9}$$

and
$$\frac{\partial^2 p_i}{\partial \beta^2} = \frac{-a_i{}^2(a_i\beta-\alpha)}{\sqrt{2\pi}} \exp[-(a_i\beta-\alpha)^2/2]$$

$$\frac{\partial^2 p_i}{\partial \beta^2} = -a_i{}^2 s_i x_i. \tag{B10}$$

Similarly for the q_j,

Let
$$\frac{\partial q_j}{\partial \alpha} = (1/\sqrt{2\pi}) \exp[-(b_j\beta-\alpha)^2/2] = y_j \tag{B11}$$

then,
$$\frac{\partial^2 q_j}{\partial \alpha^2} = \frac{\partial y_j}{\partial \alpha} = \frac{(b_j\beta-\alpha)}{\sqrt{2\pi}} \exp[-(b_j\beta-\alpha)^2/2]$$

$$\frac{\partial^2 q_j}{\partial \alpha^2} = t_j y_j, \tag{B12}$$

$$\frac{\partial^2 q_j}{\partial \alpha \partial \beta} = \frac{\partial y_j}{\partial \beta} = \frac{-b_j(b_j\beta-\alpha)}{\sqrt{2\pi}} \exp[-(b_j\beta-\alpha)^2/2]$$

$$\frac{\partial^2 q_j}{\partial \alpha \partial \beta} = -\, b_j t_j y_j, \tag{B13}$$

$$\frac{\partial q_j}{\partial \beta} = \frac{-b_j}{\sqrt{2\pi}} \exp[-(b_j\beta-\alpha)^2/2] = -\, b_j y_j, \tag{B14}$$

and
$$\frac{\partial^2 q_j}{\partial \beta^2} = \frac{b_j{}^2(b_j\beta-\alpha)}{\sqrt{2\pi}} \exp[-(b_j\beta-\alpha)^2/2]$$

$$\frac{\partial^2 q_j}{\partial \beta^2} = b_j{}^2 t_j y_j. \tag{B15}$$

Using the above results, we have

$$L\alpha = \frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha} (\ln p_i) + \sum_{j=1}^{m} \frac{\partial}{\partial \alpha} (\ln q_j)$$

$$= \sum_{i=1}^{n} \frac{1}{p_i} \frac{\partial p_i}{\partial \alpha} + \sum_{j=1}^{m} \frac{1}{q_j} \frac{\partial q_j}{\partial \alpha}$$

$$L\alpha = \sum_{j=1}^{m} (y_j/q_j) - \sum_{i=1}^{n} (x_i/p_i), \tag{B16}$$

$$L\alpha\alpha = \frac{\partial^2 L}{\partial \alpha^2} = \sum_{j=1}^{m} \frac{\partial}{\partial \alpha} \left(\frac{y_j}{q_j} \right) + \sum_{i=1}^{n} \frac{\partial}{\partial \alpha} \left(\frac{-x_i}{p_i} \right)$$

$$= \sum_{j=1}^{m} \frac{q_j \left(\frac{\partial y_j}{\partial \alpha} \right) - y_j \left(\frac{\partial q_j}{\partial \alpha} \right)}{q_j^2}$$

$$+ \sum_{i=1}^{n} \frac{p_i \left(\frac{\partial (-x_i)}{\partial \alpha} \right) + x_i \left(\frac{\partial p_i}{\partial \alpha} \right)}{p_i^2}$$

$$= \sum_{j=1}^{m} \left(\frac{t_j y_j}{q_j} - \frac{y_j^2}{q_j^2} \right) + \sum_{i=1}^{n} \left(\frac{-s_i x_i}{p_i} - \frac{x_i^2}{p_i^2} \right)$$

$$L\alpha\alpha = - \sum_{j=1}^{m} (y_j/q_j)[(y_j/q_j) - t_j] - \sum_{i=1}^{n} (x_i/p_i)[(x_i/p_i) + s_i], \tag{B17}$$

$$L\alpha\beta = \frac{\partial^2 L}{\partial\alpha\partial\beta} = \sum_{j=1}^{m} \frac{\partial}{\partial\beta} \left(\frac{y_j}{q_j} \right) - \sum_{i=1}^{n} \frac{\partial}{\partial\beta} \left(\frac{x_i}{p_i} \right)$$

$$= \sum_{j=1}^{m} \frac{q_j\left(\frac{\partial y_j}{\partial\beta} \right) - y_j\left(\frac{\partial q_j}{\partial\beta} \right)}{q_j^{\,2}}$$

$$+ \sum_{i=1}^{n} \frac{p_i\left(\frac{\partial(-x_i)}{\partial\beta} + x_i\left(\frac{\partial p_i}{\partial\beta} \right)\right.}{p_i^{\,2}}$$

$$= \sum_{j=1}^{m} \left(\frac{-b_j t_j y_j}{q_j} + \frac{b_j y_j^{\,2}}{q_j^{\,2}} \right) + \sum_{i=1}^{n} \left(\frac{a_i s_i x_i}{p_i} + \frac{a_i x_i^{\,2}}{p_i^{\,2}} \right)$$

$$L\alpha\beta = \sum_{j=1}^{m} b_j(y_j/q_j)[(y_j/q_j) - t_j]$$

$$+ \sum_{i=1}^{n} a_i(x_i/p_i)[(x_i/p_i) + s_i]. \qquad \text{(B18)}$$

Similarly,

$$L\beta = \frac{\partial L}{\partial\beta} = \sum_{i=1}^{n} \frac{\partial}{\partial\beta} (\ell n\ p_i) + \sum_{j=1}^{m} \frac{\partial}{\partial\beta} (\ell n\ q_j)$$

$$= \sum_{i=1}^{n} \frac{1}{p_i} \left(\frac{\partial p_i}{\partial\beta} \right) + \sum_{j=1}^{m} \frac{1}{q_j} \left(\frac{\partial q_j}{\partial\beta} \right)$$

$$L\beta = \sum_{i=1}^{n} a_i(x_i/p_i) - \sum_{j=1}^{m} b_j(y_j/q_j), \qquad \text{(B19)}$$

$$L\beta\beta = \frac{\partial^2 L}{\partial \beta^2} = \sum_{i=1}^{n} \frac{\partial}{\partial \beta} \left(\frac{a_i x_i}{p_i} \right) - \sum_{j=1}^{m} \frac{\partial}{\partial \beta} \left(\frac{b_j y_j}{q_j} \right)$$

$$= \sum_{i=1}^{n} \frac{a_i [p_i (\frac{\partial x_i}{\partial \beta}) - x_i (\frac{\partial p_i}{\partial \beta})]}{p_i^2}$$

$$- \sum_{j=1}^{m} \frac{b_j [q_j (\frac{\partial y_j}{\partial \beta}) - y_j (\frac{\partial q_j}{\partial \beta})]}{q_j^2}$$

$$= \sum_{i=1}^{n} a_i [(\frac{-a_i s_i x_i}{p_i}) - (\frac{a_i x_i^2}{p_i^2})]$$

$$- \sum_{j=1}^{m} b_j \; [(\frac{-b_j t_j y_j}{q_j}) - (\frac{-b_j y_j^2}{q_j^2})]$$

$$L\beta\beta = - \sum_{j=1}^{m} b_j^2 (y_j/q_j)[(y_j/q_j) - t_j]$$

$$- \sum_{i=1}^{n} a_i^2 (x_i/p_i)[(x_i/p_i) + s_i]. \qquad \text{(B20)}$$

APPENDIX C. INPUT NEEDED AND PROGRAM LISTING

APPENDIX C. INPUT NEEDED AND PROGRAM LISTING

The A. R. DiDonato and M. P. Jarnagin, Jr. maximum likelihood estimating program is available on the Ballistic Research Laboratory's CDC computer system.

Prior to executing the program, a data file must be created and stored on MFA (mainframe A). A list of the parameters that must be read in as input and the format and instructions needed to create the data file are provided here.

The three (3) card jobstream that must be created and submitted to MFZ (mainframe Z) in order to run the program is also given.

It is important to remember that the data file must be made accessible to the MFZ while in the interactive facility (IAF) mode.

The following list briefly describes each of the parameters which must be read into the program prior to execution and which also comprises the calling sequence for the main computing subroutine EPPA.

CALL EPPA (IDENT,K,L,ALPHO,BETO,FNA,A,FNB,B)

IDENT - Array dimensioned at 8 locations with up to 80 characters allowed per job. The Hollerith character content of IDENT is printed on the top line of the output. Note: IDENT is not printed when the nonparametric procedure is employed.

K- Twice n, where n is the number of numerically different a_i's (responses). $n \geq 2$.

L- Twice m, where m is the number of numerically different b_j's (nonresponses). $m \geq 2$.

ALPHO,BETO - User supplied starting values α_o, β_o, respectively, to the algorithm. Setting BETO ≤ 0 allows the user to have the algorithm compute the starting values $\alpha_o = \mu_o/\sigma_o$ and $\beta_o = 1/\sigma_o$.

FNA,A - Arrays dimensioned at $k = 2 \cdot n$. FNA(i) specifies the number of A(i) values used, i = 1,2, ...,n. FNA(n+1), FNA(n+2), ..., FNA(2n) and A(n+1), A(n+2), ..., A(2n) is used by EPPA as workspace.

FNB,B - Arrays dimensioned at $\ell = 2 \cdot m$. FNB(i) specifies the number of B(i) values used, i = 1,2, ...,m. FNB(m+1), FNB(m+2), ...FNB(2m) and B(m+1), B(m+2), ..., B(2m) is used by EPPA as workspace.

STEP 1. CREATING THE DATA FILE

Data should be created and stored as an MFA permanent file.

Card 1	Columns	Format
IDENT	1-80	8A10
Card 2 2 inputs		
K	1-5	I5
L	6-10	I5
Card 3 2 inputs		
ALPHO	1-10	F10.2
BETO	11-20	F10.2

Cards 4,5

Input may consist of a maximum of 8 data points per card. More than one data card may be required to input each parameter. These cards contain data read in with an F10.2 format.

Columns	1-10	11-20	....	71-80
Card 4	FNA(1)	FNA(2)		FNA(8)
Card 5	A(1)	A(2)		A(8)

Cards 6,7

Input may consist of a maximum of 8 data points per card. More than one data card may be required to input each parameter. These cards contain data read in with an F10.2 format.

Columns	1-10	11-20	....	71-80
Card 6	FNB(1)	FNB(2)		FNB(8)
Card 7	B(1)	B(2)		B(8)

STEP 2.

Prior to running the program, the data file must be made accessible to the MFZ while in IAF (/) mode with

PERMIT, PFN, MFZ

where

PFN = filename under which the input data is created

STEP 3.

To run the program, create and submit the following 3 card jobstream to MFZ.

JOBNAME, STMFZ.

ACCOUNT, XXXXXX.

BEGIN, NMLEX, NMLEX, PFI=_____, UN=_____, (RJE=RJE_____).

where

PFI = file name under which the input data is stored,

UN = user name identification for above file,

RJE = RJEXXXX where XXXX is a 4-digit code designating a particular RJE as the output device; if omitted, the central site serves as the output destination.

```
      PROGRAM DOMLE (INPUT,OUTPUT,TAPE8)
      DIMENSION IDENT(8)
      DIMENSION FNA(100), A(10     FNB(100), B(100)
      REAL IDENT
      READ (8,10) IDENT
      READ (8,15) K,L
      READ (8,20) ALPHO, BETO
      N=K/2
      M=L/2
      READ (8,25) (FNA(I),I=1,N)
      READ (8,25) (A(I),I=1,N)
      READ (8,25) (FNB(I),I=1,M)
      READ (8,25) (B(I),I=1,M)
      CALL EPPA (IDENT,K,L,ALPHO,BETO,FNA,A,FNB,B)
   10 FORMAT (8A10)
   15 FORMAT (2I5)
   20 FORMAT (2F10.2)
   25 FORMAT (8F10.2)
      STOP
      END
      SUBROUTINE EPPA (IDENT,NM,MM,ALPHA3,BETA3,FNA,A,FNB,B)
C
C EPPA - EXPLORATORY PROGRAM FOR PROBIT ANALYSIS.
C INPUT TO SUBROUTINE EPPA IS TO BE SPECIFIED AS TWO SETS OF
C REAL NUMBERS. AT THE OUTSET, THE PROGRAM IS CALLED UPON TO
C INSURE THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE
C EXISTENCE OF A UNIQUE POINT AT WHICH THE LIKELIHOOD
C FUNCTION ATTAINS A MAXIMUM ARE SATISFIED. IF THE CONDITIONS
C ARE SATISFIED, THE PROGRAM PROCEEDS TO OBTAIN INITIAL
C APPROXIMATIONS FOR ALPHA AND BETA, AND THE NEWTON-RAPHSON
C PROCEDURE IS EMPLOYED TO OBTAIN THE KTH APPROXIMATIONS TO
C ALPHA AND BETA. UPON CONVERGENCE, THE ROUTINE PROCEEDS WITH
C THE CALCULATIONS OF THE COVARIANCE MATRIX ELEMENTS OF A.
C IF EITHER OF THE CONDITIONS IS NOT MET, A NONPARAMETRIC PROCEDURE
C IS CALLED UPON TO CALCULATE ESTIMATES OF MU AND SIGMA.
C
```

```
      COMMON /ZZZZ/ BETA, BETA0, ALPHA, QG
      COMMON /DANDE/ EP1, EP2, LIMIT, NC
      DIMENSION A(NM), FNA(NM), B(MM), FNB(MM)
      DIMENSION IDENT(8)
      DIMENSION G(100), F(100)
      DIMENSION GGSAV(100)
      EQUIVALENCE (ALPHA0,ALPHA,A1), (BETA0,B1)
      EQUIVALENCE (R,A00), (S,A0S), (T,ASS)
      REAL MU0
      DATA EP1,EP2,LIMIT,NC /2.5E-4,5.E-4,100, 4/
      ALPHA0=ALPHA3
      BETA0=BETA3
      K=1
      ML=MM/2
      NL=NM/2
      SUMA=0.0
      SUMB=0.0
      MIN=MIN0(ML,NL)
      C=0.0
      FNLP=0.0
      DO 5 I=1,NL
      CALL SORT (A,NL,FNA)
      IF (FNA(I).LE.0.0) FNA(I)=1.
      FNLP=FNLP+FNA(I)
      SUMA=SUMA+A(I)*FNA(I)
      C=C+A(I)*A(I)*FNA(I)
      AMIN=A(1)
    5 CONTINUE
      NTOTA=FNLP+0.00001
      FMLP=0.0
      DO 10 I=1,ML
      CALL SORT (B,ML,FNB)
      IF (FNB(I).LE.0.0) FNB(I)=1.
      FMLP=FMLP+FNB(I)
      C=C+B(I)*B(I)*FNB(I)
      SUMB=SUMB+B(I)*FNB(I)
```

```
      BMAX=B(ML)
   10 CONTINUE
      MTOTB=FMLP+0.00001
      FNLML=FNLP+FMLP
      IF (AMIN.GE.BMAX) GO TO 15
      CC=(SUMA+SUMB)/FNLML
      SUMA=SUMA/FNLP
      SUMB=SUMB/FMLP
      IF (SUMA.GT.SUMB) GO TO 35
      GO TO 25
   15 PRINT 20
   20 FORMAT (49H0MINIMUM A IS GREATER THAN OR EQUAL TO MAXIMUM B.)
      CALL AVERAGE (A,NL,FNA,B,ML,FNB,NTOTA,MTOTB)
      GO TO 315
   25 PRINT 30
   30 FORMAT (41H0AVERAGE A IS NOT GREATER THAN AVERAGE B.)
      CALL AVERAGE (A,NL,FNA,B,ML,FNB,NTOTA,MTOTB)
      GO TO 315
   35 IF (BETA3.GT.0.) GO TO 40
      SIGMA0=C/FNLML-CC*CC
      SIGMA0=SQRT(SIGMA0)
      MU0=(SUMA+SUMB)/2.0
      ALPHA0=MU0/SIGMA0
      BETA0=1.0/SIGMA0
      GO TO 45
   40 SIGMA0=1.0/BETA0
      MU0=ALPHA0*SIGMA0
   45 PRINT 50, IDENT
   50 FORMAT (1H2,8A10)
      ALPH=ALPHA0
      BET=BETA0
      DO 55 K=1,LIMIT
      QQ=1.0
      CONST=1.
      CALL LCUM (SUM1,SUM2,[illegible],A,FNA,CONST,TONEX,TTWOX,THREEX,TFOURX,TFIV
     1EX)
```

```
      CONST=-1.
      CALL LCOM (SUM3,SUM4,MM,B,FNB,CONST,TONEY,TTWOY,THREEY,TFOURY,TFIV
     1EY)
      FLB=SUM1-SUM3
      FLAB=TFIVEY-TFOURY+TFIVEX+TFOURX
      FLBB=SUM4-SUM2
      FLA=TONEY-TONEX
      FLAA=TTWOY-THREEY-TTWOX-THREEX
      DELTAO=FLAA*FLBB-FLAB*FLAB
      G(K)=(FLB*FLAB-FLA*FLBB)/DELTAO
      F(K)=(FLA*FLAB-FLB*FLAA)/DELTAO
      BETAO=BETAO+F(K)
      ALPHAO=ALPHAO+G(K)
      SUM3=1.0/BETAO
      SUM4=ALPHAO/BETAO
      QQSAV(K)=QQ
      IF (ABS(G(K)).GE.ABS(EP1*ALPHAO).OR.ABS(F(K)).GE.ABS(EP2*BETAO)) G
     1O TO 55
      GO TO 65
   55 CONTINUE
      PRINT 60, LIMIT
   60 FORMAT (66H0THE DESIRED IMPROVEMENT IN ALPHA AND BETA HAS NOT BEEN
     1 MADE AFTER,I4,12H ITERATIONS.)
      K=LIMIT
   65 R=0.0
      S=0.0
      T=0.0
      DO 70 J=1,NL
      L=NL+J
      IF (FNA(L).EQ.0.) GO TO 70
      IF (A(L).EQ.0.0) GO TO 70
      IF (FNA(L).EQ.A(L)) GO TO 70
      C=FNA(J)*FNA(L)/(1./A(L)-1./FNA(L))
      U=A(J)*B1-A1
      V=U*C
      R=R+C
```

```
      S=S+V
      T=T+U*V
   70 CONTINUE
      DO 75 J=1,ML
      L=ML+J
      IF (FNB(L).EQ.0.0) GO TO 75
      IF (B(L).EQ.0.0) GO TO 75
      IF (FNB(L).EQ.B(L)) GO TO 75
      C=FNB(J)*FNB(L)/(1./B(L)-1./FNB(L))
      U=B(J)*B1-A1
      V=U*C
      R=R+C
      S=S+V
      T=T+U*V
   75 CONTINUE
      C=B1*B1
      R=C*R
      S=C*S
      T=C*T
      U=AUU*ASS-AUS*AUS
      RDL=1./U
      AUUU=ASS*RDL
      AUSS=AUU*RDL
      AUUS=-AUS*RDL
      STDM=SQRT(AUUU)
      STDS=SQRT(AUSS)
      PRINT 80
   80 FORMAT (6H0 A(I),17X,4HB(J))
      KK=MINK
      PRINT 85, A(1),B(1),SUM4,SUM3
   85 FORMAT (4X,3H1) ,E11.5,7X,3H1) ,E11.5,7X,3HMU=,E20.14,3X,6HSIGMA=,
     1E20.14)
      DO 115 I=2,MINK
      IF (I.EQ.3) GO TO 95
      IF (I.EQ.4) GO TO 105
      PRINT 90, I,A(I),I,B(I)
```

```
 90 FORMAT (1H ,I4,2H) ,E11.5,4X,I4,2H) ,E11.5)
    GO TO 115
 95 PRINT 100, A(I),B(I),AUUU,AUUS
100 FORMAT (4X,3H3) ,E11.5,7X,3H3) ,E11.5,7X,17HCOVARIANCE MATRIX,2E22
   1.14)
    GO TO 115
105 PRINT 110, A(I),B(I),AUUS,AUSS
110 FORMAT (4X,3H4) ,E11.5,7X,3H4) ,E11.5,24X,2E22.14)
115 CONTINUE
    IF (NL-ML) 155,190,120
120 MA=NL-ML
    DO 150 I=1,MA
    KK=MINK+I
    IF (KK.EQ.3) GO TO 130
    IF (KK.EQ.4) GO TO 140
    PRINT 125, KK,A(KK)
125 FORMAT (1H ,I4,2H) ,E11.5)
    GO TO 150
130 PRINT 135, A(KK),AUUU,AUUS
135 FORMAT (4X,3H3) ,E11.5,28X,17HCOVARIANCE MATRIX,2E22.14)
    GO TO 150
140 PRINT 145, A(KK),AUUS,AUSS
145 FORMAT (4X,3H4) ,E11.5,45X,2E22.14)
150 CONTINUE
    GO TO 190
155 MB=ML-NL
    DO 185 I=1,MB
    KK=MINK+I
    IF (KK.EQ.3) GO TO 165
    IF (KK.EQ.4) GO TO 175
    PRINT 160, KK,B(KK)
160 FORMAT (22X,I4,2H) ,E11.5)
    GO TO 185
165 PRINT 170, B(KK),AUUU,AUUS
170 FORMAT (25X,3H3) ,E11.5,7X,17HCOVARIANCE MATRIX,2E22.14)
    GO TO 185
```

```
175 PRINT 180, B(KK),AUUS,AUSS
180 FORMAT (25X,3H4) ,E11.5,24X,2E22.14)
185 CONTINUE
190 IF (KK.GE.3) GO TO 200
    PRINT 195, AUUU,AUUS
195 FORMAT (46X,17HCOVARIANCE MATRIX,2E22.14)
200 IF (KK.EQ.3) PRINT 205, AUUS,AUSS
205 FORMAT (63X,2E22.14)
    PRINT 210
210 FORMAT (19H0NUMBER OF A VALUES,9X,8HB VALUES)
    DO 230 I=1,MINK
    I4=FNA(I)
    I6=FNB(I)
    IF (I.EQ.1) GO TO 220
    PRINT 215, I,I4,I,I6
215 FORMAT (1H ,I4,2H) ,I5,10X,I4,2H) ,I5)
    GO TO 230
220 PRINT 225, I,I4,I,I6,STDM,STDS
225 FORMAT (1H ,I4,2H) ,I5,10X,I4,2H) ,I5,13X,11HSTD DEV MU=,E20.14,3X
   1,14HSTD DEV SIGMA=,E20.14)
230 CONTINUE
    IF (NL-ML) 235,265,250
235 DO 240 I=1,MB
    KK=MINK+I
    I4=FNB(KK)
240 PRINT 245, KK,I4
245 FORMAT (1H ,21X,I4,2H) ,I5)
    GO TO 265
250 DO 255 I=1,MA
    KK=MINK+I
    I4=FNA(KK)
255 PRINT 260, KK,I4
260 FORMAT (1H ,I4,2H) ,I5)
265 CONTINUE
    I4=FNLP
    I6=FMLP
```

```
      PRINT 270, I4,I6
  270 FORMAT (1H0,5HTOTAL,I6,15X,I6)
      IF (BETA3.LE.0.) GO TO 280
      PRINT 275, ALPH,BET,MUU,SIGMAU
  275 FORMAT (15H0ALPHAU(INPUT)=,E21.14,15H  BETAU(INPUT)=,E21.14,6H  MU
     1U=,E21.14,9H  SIGMAU=,E21.14)
      GO TO 290
  280 PRINT 285, ALPH,BET,MUU,SIGMAU
  285 FORMAT (8H0ALPHAU=,E21.14,8H  BETAU=,E21.14,8H    MUU=,E21.14,9H
     1SIGMAU=,E21.14)
  290 PRINT 295
  295 FORMAT (5H0STEP,10X,11HDELTA ALPHA,11X,10HDELTA BETA,14X,2HLW)
      DO 300 I=1,K
  300 PRINT 305, I,G(I),F(I),GGSAV(I)
  305 FORMAT (1H ,I3,2X,3E21.14)
      PRINT 310, GG,DELTAU,ALPHAU,BETAU
  310 FORMAT (9H0MAXIMUM=,E20.14,8H  DELTA=,E20.14,8H  ALPHA=,E20.14,7H
     1 BETA=,E20.14)
  315 CONTINUE
      RETURN
      END
      SUBROUTINE LCOM (SUM1,SUM2,N,A,FN,CONST,TONES,TTWOS,TTHRES,TFOUR,
     1  TFIVE)
C
C THIS SUBROUTINE CARRIES OUT THE CALCULATIONS ASSOCIATED
C WITH THE MAXIMUM LIKELIHOOD EQUATIONS AND THE SECOND
C ORDER PARTIAL DERIVATIVES OF L.
C
      COMMON /UANDE/ EP1, EP2, LIMIT, NC
      COMMON /ZZZ/ BETA, BETAU, ALPHA, GG
      COMMON /CPNDF/ ENDF
      DIMENSION RATION(5)
      DIMENSION A(N), FN(N)
      DATA (RATION(K),K=3,5)/.5,.66666666666667,.75/
      DATA SQPI/.39894228040143/
      ENDF=0.
```

```
      SUM1=0.0
      SUM2=0.0
      TONES=0.0
      TTWOS=0.0
      TTHRES=0.0
      TFOUR=0.0
      TFIVE=0.0
      N2=N/2
      DO 50 I=1,N2
      SI=A(I)*BETAO-ALPHA
      ZSI=-SI*SI/2.0
      IF (ZSI.LT.-675.82) GO TO 40
      A(N2+I)=SQPI*EXP(ZSI)
    5 CONTINUE
      IF (ABS(SI).GT.6.0) GO TO 30
      IGO=1
      IF (CONST.GE.0.) GO TO 10
      CHECK=PNDF(-SI,0)
      GO TO 15
   10 CHECK=PNDF(SI,0)
   15 QQ=QQ*CHECK**FN(I)
      TONE=A(N2+I)/CHECK
      FN(N2+I)=TONE
      TONE=FN(I)*TONE
   20 TEMP=A(I)*TONE
      SUM1=SUM1+TEMP
      TTWO=TONE*SI
      TTHREE=TONE*FN(N2+I)
      IF (IGO.EQ.1) GO TO 25
      IF (IGO.EQ.2) GO TO 35
   25 SUM2=A(I)*A(I)*(TTWO+CONST*TTHREE)+SUM2
      GO TO 35
   30 CONSTS=CONST*SI
      IF (CONSTS.GE.0.0) GO TO 45
      QQ=0.0
      TONE=PNDF(SI,IFIX(CONST-1.))
```

```
      FN(N2+I)=TONE
      TONE=FN(I)*TONE
      SUM2=SUM2-ENUF*A(I)*A(I)*CONST*FN(I)
      IGO=2
      GO TO 20
   35 TONES=TONES+TONE
      TTHRES=TTHRES+TTHREE
      TTWOS=TTWOS+TTWO
      TFOUR=TFOUR+TEMP*SI
      TFIVE=TFIVE+TTHREE*A(I)
      GO TO 50
   40 CONTINUE
      A(N2+I)=0.
      GO TO 5
   45 FN(N2+1)=0.0
   50 CONTINUE
      RETURN
      END
      FUNCTION PNUF (X,IW)
C
C THIS SUBROUTINE CALCULATES THE PROBABILITY INTEGRAL, ASSUMING
C A CUMULATIVE NORMAL DISTRIBUTION.
C
      COMMON /CPNUF/ ENUF
      LOGICAL L1, L2
      DIMENSION A(5), B(5), AA(9), BB(9), AAA(6), BBB(6)
      DATA (A(I),I=1,5) /.18577770618460E-0,.31611237438706E+1,
     1.11386415415105E+3,.37748523768530E+3,.32093775891385E+4/
      DATA (B(I),I=1,5) /.1            E+1,.23601290952344E+2,
     1.24402463793444E+3,.12826165260774E+4,.28442368334392E+4/
      DATA (AA(I),I=1,9) /.21531153547440E-7,.56418849698867E+0,
     1.88831497943886E+1,.66119190637142E+2,.29863513819740E+3,
     2.88195222124177E+3,.17120476126341E+4,.20510783778261E+4,
     3.12303393547980E+4/
      DATA (BB(I),I=1,9) /.1            E+1,.15744926110710E+2,
     1.11769395089131E+3,.53718110186201E+3,.16213895745667E+4,
```

```
 2.32907992357335E+4,.43626190901432E+4,.34393676741437E+4,
 3.12303393548037E+4/
  DATA (AAA(I),I=1,6) /-.16315387137302E-1,-.30532660349612E-0,
 1-.36034489994980E-0,-.12578172611123E-0,-.16063785148742E-1,
 2-.65874916152984E-3/
  DATA (BBB(I),I=1,6) /.1              E+1,.25660520192269E+1,
 1.18729528499235E+1,.52790510295143E+0,.60518341312441E-1,
 2.23352049762687E-2/
  DATA C0,C1,C2,C3 /0.,1.,2.,.5/
  DATA C4,C5,C6 /1.4142135623731,.56418958354776,1.77245385090551/
  XSAV=X
  XA=ABS(X)
  Y=X/C4
  YA=ABS(Y)
  S=Y*Y
  PA=C0
  PB=C0
  IF (YA.GT.C3) GO TO 15
  DO 5 I=1,5
  PA=PA*S+A(I)
5 PB=PB*S+B(I)
  T=(PA/PB)*Y/C2
  IF (IG.NE.0) GO TO 10
  PNDF=T+C3
  RETURN
10 PNDF=C3-T
  RETURN
15 L1=X.GT.C0.AND.IG.EQ.0.OR.X.LT.C0.AND.IG.NE.0
  IF (YA.GE.4.) GO TO 25
  DO 20 I=1,9
  PA=PA*YA+AA(I)
20 PB=PB*YA+BB(I)
  T=PA/PB
  GO TO 35
25 L2=XA.GT.8.
  IF (L1.AND.L2) GO TO 40
```

```
      Y=C1/S
      DO 30 I=1,6
      PA=PA*Y+AAA(I)
   30 PB=PB*Y+BBB(I)
      X=PA/PB
      T=X*Y
      IF (L2) GO TO 45
      T=(T+C5)/YA
   35 PNDF=EXP(-S)*T/C2
      IF (L1) PNDF=C1-PNDF
      X=XSAV
      RETURN
   40 PNDF=C1
      RETURN
   45 Y=T*C6+C1
      PNDF=XA/Y
      ENDF=X*C6*C2/(Y*Y)
      X=XSAV
      RETURN
      END
      SUBROUTINE SORT (A,N,FN)
C
C THIS SUBROUTINE SORTS EACH INPUT ARRAY ACCORDING TO LEVELS,
C WITH THE LOWEST LEVEL OF EACH ARRAY OCCUPYING THE FIRST
C POSITION.
C
      DIMENSION A(N), FN(N)
      NK=N-1
      DO 15 I=1,NK
      J=I+1
      DO 10 K=J,N
      IF (A(I).GT.A(K)) GO TO 5
      GO TO 10
    5 SLIST=A(I)
      A(I)=A(K)
      A(K)=SLIST
```

```
      SVAL=FN(I)
      FN(I)=FN(K)
      FN(K)=SVAL
   20 CONTINUE
   15 CONTINUE
      RETURN
      END
      SUBROUTINE AVERAGE (A,N,FN,B,M,FM,NTOTA,NTOTB)
C
C THIS SUBROUTINE COMPUTES THE NONPARAMETRIC ESTIMATES
C OF MU AND SIGMA WHEN EITHER OF THE ABOVE RESTRICTIONS
C IS NOT MET, USING THE METHOD OF AVERAGING CERTAIN LEVELS.
C TO ESTIMATE MU, THE ROUTINE AVERAGES THE LOWEST RESPONSE
C AND THE HIGHEST NON-RESPONSE FIRST. THEN IT TAKES THE
C FIRST TWO LOWEST RESPONSES AND THE FIRST TWO HIGHEST
C NON-RESPONSES; AND SO ON UNTIL ALL EVEN NUMBER COMBINATIONS,
C UP TO A MAXIMUM OF SIX LEVELS, ARE  EXHAUSTED. THE ESTIMATE
C OF SIGMA IS FOUND BY TAKING 1/2 TIMES THE RANGE, WHERE THE
C RANGE = MAX(B0(M),A0(I)) - MIN(B0(J),A0(1)). B0(M) IS THE
C HIGHEST NON-RESPONSE AND A0(1) IS THE LOWEST RESPONSE.
C THE ESTIMATING PROCEDURE FOR MU AND SIGMA HAS BEEN FACILITATED
C BY EXPANDING THE TWO INPUT ARRAYS TO INCLUDE ALL RESPONSES AND
C ALL NON-RESPONSES.
C
      DIMENSION A(N), B(M), FN(N), FM(M), A0(100), B0(100)
      REAL MU0
      REAL MU
      PRINT 5
    5 FORMAT (10H0RESPONSES)
      KOUNT=0
      DO 25 II=1,N
      KOUNT=KOUNT+1
      A0(KOUNT)=A(II)
      PRINT 10, KOUNT,A0(KOUNT)
   10 FORMAT (2X,I3,2H) ,E11.5)
      IF (FN(II).EQ.1) GO TO 25
```

```
      NFN=FN(II)
      DO 20 KK=2,NFN
      KOUNT=KOUNT+1
      A0(KOUNT)=A(II)
      PRINT 15, KOUNT,A0(KOUNT)
   15 FORMAT (2X,I3,2H) ,E11.5)
   20 CONTINUE
   25 CONTINUE
      PRINT 30
   30 FORMAT (14H0NON-RESPONSES)
      KOUNT=0
      DO 50 JJ=1,M
      KOUNT=KOUNT+1
      B0(KOUNT)=B(JJ)
      PRINT 35, KOUNT,B0(KOUNT)
   35 FORMAT (2X,I3,2H) ,E11.5)
      IF (FM(JJ).EQ.1) GO TO 50
      MFM=FM(JJ)
      DO 45 KK=2,MFM
      KOUNT=KOUNT+1
      B0(KOUNT)=B(JJ)
      PRINT 40, KOUNT,B0(KOUNT)
   40 FORMAT (2X,I3,2H) ,E11.5)
   45 CONTINUE
   50 CONTINUE
      MU0=0.0
      SIGMA=0.0
      I=1
      J=MTOTB
      KOUNT=1
      NRNDS=NTOTA
      IF (NTOTA.GE.MTOTB) NRNDS=MTOTB
      DO 85 K=1,NRNDS
      MU0=MU0+(A0(I)+B0(J))/2.0
      L=KOUNT*2
      W=KOUNT
```

```
      IF (B0(MTOTB).GT.A0(I)) GO TO 55
      RRANGE=A0(I)
      GO TO 60
   55 RRANGE=B0(MTOTB)
   60 IF (B0(J).LT.A0(1)) GO TO 65
      SRANGE=A0(1)
      GO TO 70
   65 SRANGE=B0(J)
   70 MU=MU0/M
      SIGMA=(RRANGE-SRANGE)/2.0
      IEST=KOUNT
      PRINT 75, IEST,L
   75 FORMAT (10H0ESTIMATE ,I1,31H FOR MU AND SIGMA WAS BASED ON ,I1,6H
     1ROUNDS.)
      PRINT 80, MU,SIGMA
   80 FORMAT (1X,4HMU= ,E20.14,5X,7HSIGMA= ,E20.14)
      KOUNT=KOUNT+1
      IF (KOUNT.GT.3) GO TO 90
      I=I+1
      J=J-1
   85 CONTINUE
   90 CONTINUE
      RETURN
      END
```

APPENDIX D. COMPUTER OUTPUT FOR EXAMPLES

```
TEST DATA - EXAMPLE 1

 A(I)                   B(J)
   1) 0.94300e+03          1) 0.92400e+03      MU=0.94882263183594e+03   SIGMA=0.29537014007568e+02
   2) 0.94700e+03          2) 0.93100e+03
   3) 0.96000e+03          3) 0.94200e+03      COVARIANCE MATRIX  0.18350585937500e+03 -0.50653255462646e+02
   4) 0.97300e+03          4) 0.97600e+03                        -0.50653255462646e+02  0.35894644165039e+03
   5) 0.96900e+03
   6) 0.10080e+04

NUMBER OF A VALUES         B VALUES
   1)      1               1)     1        .   STD DEV MU=0.13546433448792e+02   STD DEV SIGMA=0.18945 32797241e+02
   2)      1               2)     1
   3)      1               3)     1
   4)      1               4)     1
   5)      1
   6)      1

TOTAL      6                      4

ALPHAO= 0.37567100524902e+02  BETAO= 0.39270456850331e-01    MUO= 0.95662500000000e+03  SIGMAO= 0.25464435577393e+02

STEP             DELTA ALPHA             DELTA BETA              L*
  1   -0.62739062309265e+01-0.62746661715209e-02 0.45628538355231e-02
  2    0.82161611319588e+00 0.85129414219409e-03 0.56444550364398e-02
  3    0.82587422803044e-02 0.86314603322535e-05 0.56499624624848e-02
  4    0.10214964277111e-03 0.10845491971168e-06 0.56499508209527e-02

MAXIMUM=0.56499508209527e-02  DELTA=0.10102000000000e+05  ALPHA=0.32123172760010e+02  BETA=0.3335825662613e-01
```

OUTPUT FOR EXAMPLE 1

```
MINIMUM A IS GREATER THAN OR EQUAL TO MAXIMUM B.

RESPONSES
   1) 0.96600e+03
   2) 0.97000e+03
   3) 0.97300e+03
   4) 0.98200e+03

NON-RESPONSES
   1) 0.94400e+03
   2) 0.94900e+03
   3) 0.96100e+03

ESTIMATE 1 FOR MU AND SIGMA WAS BASED ON 2 ROUNDS.
MU= 0.96350000000000e+03     SIGMA= 0.25000000000000e+01

ESTIMATE 2 FOR MU AND SIGMA WAS BASED ON 4 ROUNDS.
MU= 0.96150000000000e+03     SIGMA= 0.10500000000000e+02

ESTIMATE 3 FOR MU AND SIGMA WAS BASED ON 6 ROUNDS.
MU= 0.96050000000000e+03     SIGMA= 0.14500000000000e+02
```

OUTPUT FOR EXAMPLE 2

```
TEST DATA - EXAMPLE 3

 A(I)                    B(J)
   1) 0.12960e+04          1) 0.12950e+04      MU=0.13178927001953e+04   SIGMA=0.26016082763672e+02
   2) 0.13100e+04          2) 0.12980e+04
   3) 0.13110e+04          3) 0.13010e+04      COVARIANCE MATRIX  0.69047570800781e+03  0.11052309570313e+04
                           4) 0.13030e+04                         0.11052309570313e+04  0.21421496582031e+04
                           5) 0.13040e+04
                           6) 0.13070e+04
                           7) 0.13140e+04

 NUMBER OF A VALUES          B VALUES
   1)     1                1)     1            STD DEV MU=0.26276905059814e+02   STD DEV SIGMA=0.46283363342285e+02
   2)     1                2)     1
   3)     1                3)     1
                           4)     1
                           5)     1
                           6)     1
                           7)     1

 TOTAL      3

 ALPHAO= 0.21022386169434e+03  BETAO= 0.16116459667683e+00   MUO= 0.13044047851562e+04  SIGMAO= 0.62048368453979e+01

 STEP          DELTA ALPHA             DELTA BETA                L*
   1   -0.16544367980957e+03-0.12724642455578e+00 0.22057366732042e-03
   2    0.58920261955261e+01 0.45237513259053e-02 0.26225503534079e-02
   3   -0.53643970750272e-02-0.41598063944546e-05 0.26311944238842e-02

 MAXIMUM=0.26311944238842e-02  DELTA=0.12760000000000e+04  ALPHA=0.50656845092773e+02  BETA=0.38437761366367e-01
```

OUTPUT FOR EXAMPLE 3a

```
TEST DATA - EXAMPLE 3

A(I)                     B(J)
  1) 0.12960e+04           1) 0.12950e+04       MU=0.13178924560547e+04   SIGMA=0.26015707015991e+02
  2) 0.13100e+04           2) 0.12980e+04
  3) 0.13110e+04           3) 0.13010e+04       COVARIANCE MATRIX  0.69044287109375e+03  0.11051765136719e+04
                           4) 0.13030e+04                          0.11051765136719e+04  0.21420573730469e+04
                           5) 0.13040e+04
                           6) 0.13070e+04
                           7) 0.13140e+04

NUMBER OF A VALUES          B VALUES            STD DEV MU=0.26276279449463e+02   STD DEV SIGMA=0.46282367706299e+02
  1)     1                 1)     1
  2)     1                 2)     1
  3)     1                 3)     1
                           4)     1
                           5)     1
                           6)     1
                           7)     1

TOTAL    3                        7

ALPHAO(INPUT)= 0.26000000000000e+02  BETAO(INPUT)= 0.60000000521541e-02  MUO= 0.43333334960937e+04  SIGMAO= 0.16666567175293e+03

STEP        DELTA ALPHA            DELTA BETA              L*
  1  -0.34742853886719e+02-0.12563775293529e-01 0.00000000000000e+00
  2   0.52442787170410e+02 ..39706252515316e-01 0.45738619519398e-03
  3   0.68673543693425e+01 0.52271853201091e-02 0.26043991092592e-02
  4   0.90247547139835e-01 0.63627181452944e-04 0.26311923284084e-02
  5   0.34558772313176e-04 0.25777262990800e-07 0.26311862748116e-02

MAXIMUM=0.26311862748116e-02  DELTA=0.12800000000000e+04  ALPHA=0.50657566070557e+02  BETA=0.38438316434622e-01
```

OUTPUT FOR EXAMPLE 3b

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